Journal of the Egyptian Mathematical Society (2016) **24**, 638–643

Egyptian Mathematical Society
Journal of the Egyptian Mathematical Society
www.etms-eg.org
www.elsevier.com/locate/joems

Original Article

A numerical technique for solving fractional optimal control problems and fractional Riccati differential equations

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Received 12 October 2014; revised 6 August 2015; accepted 15 December 2015
Available online 20 April 2016

Keywords

Fractional derivative;
Fractional optimal control problem;
Caputo fractional derivative

Abstract In the present paper, we apply the Bezier curves method for solving fractional optimal control problems (OCPs) and fractional Riccati differential equations. The main advantage of this method is that it can reduce the error of the approximate solutions. Hence, the solutions obtained using the Bezier curve method give good approximations. Some numerical examples are provided to confirm the accuracy of the proposed method. All of the numerical computations have been performed on a PC using several programs written in MAPLE 13.

2010 Mathematics Subject Classification: 65D07; 65D25; 65D99

1. Introduction

A tremendous use of the fractional calculus is in basic sciences and engineering, see e.g. [1–8]. Recently, the applications have included solving various classes of nonlinear fractional differential equations numerically (see for examples Refs. [1,7] and the references therein). The Adomian decomposition method is a approach to solve the linear/nonlinear systems of fractional differential equations which gives numerical answers to any order of desired accuracy (see [9–12]). Jafari et al. [13] introduced a modified variational iteration method (MVIM) for solving Riccati differential equations and the fractional Riccati differential equation.

In this paper, we focus on fractional Riccati differential, Riccati type differential-difference equation and optimal control problems with the quadratic performance index and the dynamic system with the Caputo fractional derivative. The problem can be solved without using Hamiltonian formulas. Our tool for this aim is the Bezier curves method. There are many papers deal with the Bezier curves. Zheng et al.

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Peer review under responsibility of Egyptian Mathematical Society.

Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.joems.2015.12.003

[14] proposed the use of control points of the Bernstein–Bezier form for solving differential equations numerically and also Evrenosoglu and Somali [15] used this approach for solving singular perturbed two points boundary value problems. Also the Bezier control points method is used for solving delay differential equation (see [16]). Some other applications of the Bezier functions and control points are found in [17]. In the present work, we suggest a technique similar to the one which was used in [17] for solving fractional optimal control problems and fractional Riccati differential equations.

This study is organized as follows: In Section 2, the problem is described. Method of the Solution is explained in Section 3. In Section 4, the method is applied to a variety of examples to show efficiency and simplicity of the method. Finally, Section 5 will give a conclusion briefly.

2. Problem statement

The fractional Riccati differential with respect the time is governed by the equation given below

$$D_*^\alpha y(t) = A(t) + B(t)y(t) + C(t)y^2(t), \tag{1}$$

where $A(t)$, $B(t)$ and $C(t)$ denote given functions and α represents describing the order of the fractional derivative. There are several definitions of a fractional derivative of order $\alpha > 0$. For example, the Riemann–Liouville integral operator of order α is defined by (see [2])

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)\, dt,\ \alpha > 0,\ x > 0, \tag{2}$$

and its fractional derivative of order $\alpha \geq 0$ is

$$D^\alpha f(x) = \frac{d^m}{dx^m}(I^{m-\alpha} f(x)),\ \textit{with a suitable integer } m, \tag{3}$$

The Riemann–Liouville integral operator plays an important role in the development of the theory of fractional derivatives and integrals. However, it has some disadvantages for treating fractional differential equations with initial and boundary conditions. Therefore, we adopt here the Caputo definition, which is a modification of the Riemann–Liouville definition (see [1,2,7]):

$$D_*^\alpha f(x) = I^{m-\alpha}\left(\frac{d^m}{dx^m} f(x)\right), \tag{4}$$

where $m \in N$; $m-1 < \alpha \leq m$. The Caputo fractional derivative first computes an ordinary derivative followed by a fractional integral to achieve the desired order of fractional derivative. We mention that the Riemann–Liouville fractional derivative is computed in the reverse order. We have chosen to use the Caputo fractional derivative because it allows traditional (integer order) initial and boundary conditions to be included in the formulation of the problem, but for homogeneous initial conditions assumption, these two operators coincide. For more details on the geometric and physical interpretation of fractional derivatives of both the Riemann–Liouville and Caputo types, see Podlubny [18].

3. Method of the solution

Our strategy is using Bezier curves to approximate the solutions $y(t)$ by $v(t)$, where $v(t)$ is given below. Define the Bezier polynomials of degree n that approximate the actions of $v(t)$ over the interval $[t_0, t_f]$ as follows:

$$v(t) = \sum_{r=0}^{n} a_r B_{r,n}\left(\frac{t-t_0}{h}\right), \tag{5}$$

where $h = t_f - t_0$, and

$$B_{r,n}\left(\frac{t-t_0}{h}\right) = \binom{n}{r}\frac{1}{h^n}(t_f - t)^{n-r}(t-t_0)^r,$$

the value $B_{r,n}(\frac{t-t_0}{h})$ is the Bernstein polynomial of degree n over the interval $[t_0, t_f]$, a_r is the control point; for $r = 0, 1, \ldots, n$. By substituting (5) in (1), one may define $R_1(t)$ for $t \in [t_0, t_f]$ as follows:

$$R_1(t) = D_*^\alpha y(t) - (A(t) + B(t)y(t) + C(t)y^2(t)). \tag{6}$$

Now, by solving (6), one can find the unknown the values a_r for $r = 0, 1, \ldots, n$.

Remark 3.1. Now, the Bezier curves method is used for solving the Riccati type differential-difference equation

$$S(t)y'(\beta_1 t + \mu_1) = A(t) + B(t)y(\beta_2 t + \mu_2) + C(t)y^2(\beta_3 t + \mu_3),\ t_0 \leq t \leq t_f,$$

with the mixed condition

$$\beta_4 y(t_0) + \beta_5 y(t_f) = \lambda,$$

where $y(t)$ is an unknown function, $S(t)$, $A(t)$, $B(t)$ and $C(t)$ are the known functions defined on the interval $t_0 \leq t \leq t_f$ and β_i, for $i = 1, 2, \ldots, 5$, μ_i, for $i = 1, 2, 3$. Also t_0 and t_f are real constants.

Remark 3.2. Without loss of generality, we take the time interval as [0, 1], since any time interval $[t_0, t_f]$ can be transferred to [0, 1] by defining $t = (t_f - t_0)z + t_0$, where now $z \in [0, 1]$.

Remark 3.3. Now, we suppose that α be a real number in (0, 1), and $F, G: [t_0, t_f] \times R^2 \to R$ be two continuously differentiable functions. A general form of fractional OCPs can be introduced as (see [19]).

$$\textit{Minimize}\quad J(x,u) = \int_{t_0}^{t_f} F(t, x(t), u(t))dt, \tag{7}$$

subject to the fractional dynamic control system

$$A\dot{x}(t) + B\,{}_{t_0}^{C}D_t^\alpha x(t) = G(t, x(t), u(t)), \tag{8}$$

and the initial condition

$$x(t_0) = x_0, \tag{9}$$

where $(A, B) \neq (0, 0)$ and x_0 is a given constant. According to discussions in [20], if (x, u) be a minimum solution of (7)–(9), then there exists a $\lambda(t)$ which (x, u, λ) satisfies

$$\begin{aligned} A\dot{\lambda}(t) - B\,{}_t D_{t_f}^\alpha \lambda(t) &= -\frac{\partial H}{\partial x}(t, x, u, \lambda),\\ A\dot{x}(t) + B\,{}_{t_0}^{C}D_t^\alpha x(t) &= -\frac{\partial H}{\partial \lambda}(t, x, u, \lambda),\\ \frac{\partial H}{\partial u}(t, x, u, \lambda) &= 0,\quad t \in [t_0, t_f],\\ x(t_0) = x_0,&\quad \lambda(t_f) = 0, \end{aligned} \tag{10}$$

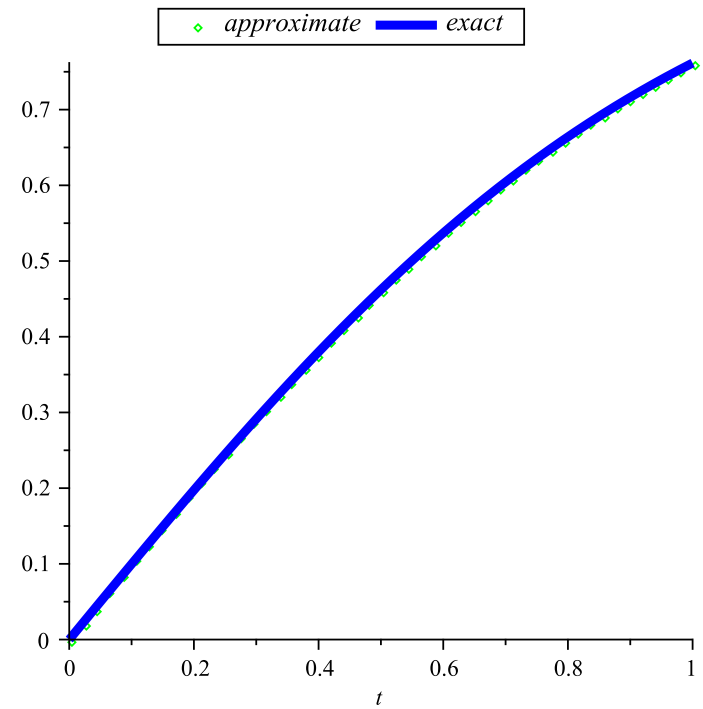

Fig. 1 The graphs of approximated and exact solution $y(t)$ for Example 1.

where

$$ {}_tD^{\alpha}_{t_f}\lambda(t) = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_t^{t_f}(\mu-t)^{n-\alpha-1}\lambda(\mu)\mu, $$

$$ {}^C_{t_0}D^{\alpha}_t x(t) = \frac{1}{\Gamma(n-\alpha)}\int_{t_0}^{t}(t-\mu)^{n-\alpha-1}x^{(n)}(\mu)\mu, $$

also, H denotes the Hamiltonian and is defined in the form of $H(t,x,u,\lambda) = F(t,x,u) + \lambda G(t,x,u)$. It should be mentioned that in practice, we obtain u in terms of λ and x from the condition $\frac{\partial H}{\partial u}(t,x,u,\lambda) = 0$. Thence, the above-mentioned system can be written in the following form

$$ \begin{aligned} A\dot{\lambda}(t) - B_t D^{\alpha}_{t_f}\lambda(t) &= M(t,x(t),\lambda(t)),\\ A\dot{x}(t) + B^C_{t_0}D^{\alpha}_t x(t) &= N(t,x(t),\lambda(t)),\\ x(t_0) = x_0,\ \lambda(t_f) &= 0, \end{aligned} \tag{11} $$

where $M(t, x(t), \lambda(t))$ and $N(t, x(t), \lambda(t))$ are known functions in terms of x and λ.

The above-mentioned fractional system contains necessary conditions for optimality of solutions of (7)–(9) (see [20]). If $F(t, x, u)$ and $G(t, x, u)$ be two convex functions in terms of x and u, then (11) contains necessary and sufficient condition for optimal solutions x^* and u^*.

Now, using Bezier curves to approximate the solutions also the variables $x(t)$ and $u(t)$ are approximated by $v(t)$ and $w(t)$ respectively where $v(t)$ and $w(t)$ are given below (see [16,17]).

$$ v(t) = \sum_{r=0}^{n} a_r B_{r,n}\left(\frac{t-t_0}{h}\right), $$

$$ w(t) = \sum_{r=0}^{n} b_r B_{r,n}\left(\frac{t-t_0}{h}\right). $$

The convergence was proved in the approximation with Bezier curves when the degree of the approximate solution, n, tends to infinity (see [17]).

4. Numerical examples

In this section, we give some computational results of numerical experiments with stated method to support our theoretical discussion.

Example 1. Consider the following fractional Riccati differential equation (see [13])

$$ \frac{d^{\alpha}y}{dt^{\alpha}} = -y^2(t) + 1,\ \ 0 < \alpha \le 1, $$
$$ y(0) = 0, $$

where the exact solution of above equations is $y(t) = \frac{e^{2t}-1}{e^{2t}+1}$ when $\alpha = 1$. By choosing $n = 7$ in the stated method and $\alpha = 0.98$, the following approximated solution can be found

$$ \begin{aligned} y(t) = {} & 8.0900837t^4 - 10.23780585t^5 - 1.635196129t^7\\ & + 6.508488745t^6 + 0.463961845t^2 - 3.427938155t^3 + t. \end{aligned} $$

The graphs of approximated and exact solution are plotted in Fig. 1.

Example 2. Consider the following fractional Riccati differential equation (see [13]):

$$ \frac{d^{\alpha}y}{dt^{\alpha}} = 2y(t) - y^2(t) + 1,\ \ 0 < \alpha \le 1, $$
$$ y(0) = 0, $$

where the exact solution was found to be of the form

$$ y(t) = 1 + \sqrt{2}\tanh\left(\sqrt{2}t + \frac{1}{2}\lg\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right), $$

when $\alpha = 1$. By choosing $n = 7$ and $\alpha = 0.98$, the following approximated solution can be found

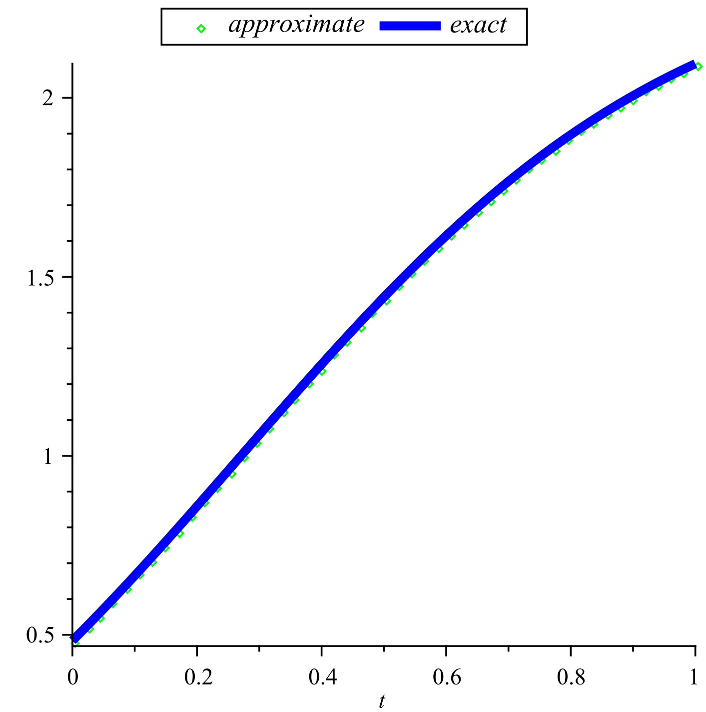

Fig. 2 The graphs of approximated and exact solution $y(t)$ for Example 2.

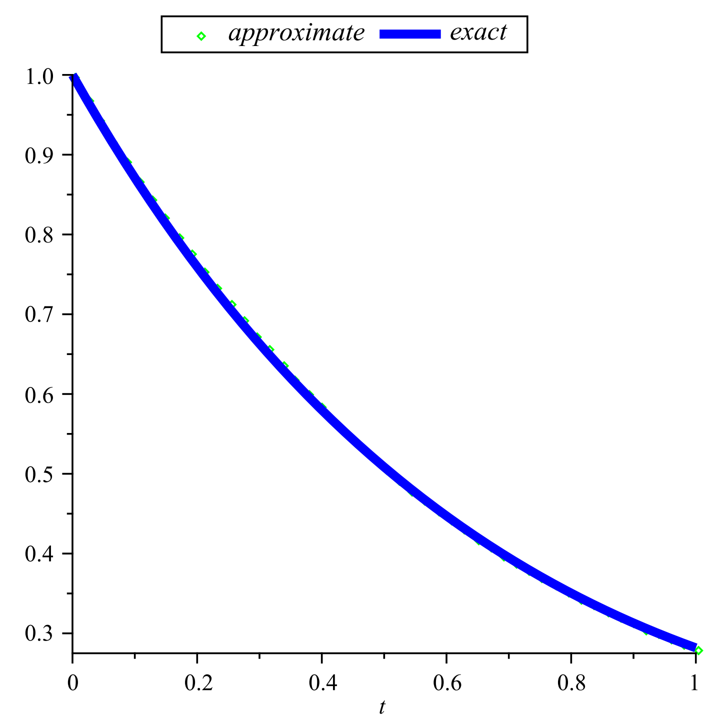

Fig. 3 The graphs of approximated and exact solution $x(t)$ for Example 3.

$$
\begin{aligned}
y(t) = {} & 1.733381253t + .4836486212 + 0.1854684368t^7 \\
& -2.540593018t^4 - 1.204247718t^6 \\
& +2.84677676t^5 + 0.8498937992t^2 \\
& -0.259041817t^3.
\end{aligned}
$$

The graphs of approximated and exact solution are plotted in Fig. 2. The modified variational iteration method (MVIM) in [13] had been presented for solving Example 2. Comparing with the modified variational iteration method for solving fractional Riccati differential equation by Bezier curves results, the results of VIM could give a more accurate approximation in a larger region (large interval) with high computation based on using the Taylor expansion but the present method does not need a large region and high computation. The use of the Bezier curves for this problem is a novel idea. Although the method is very easy

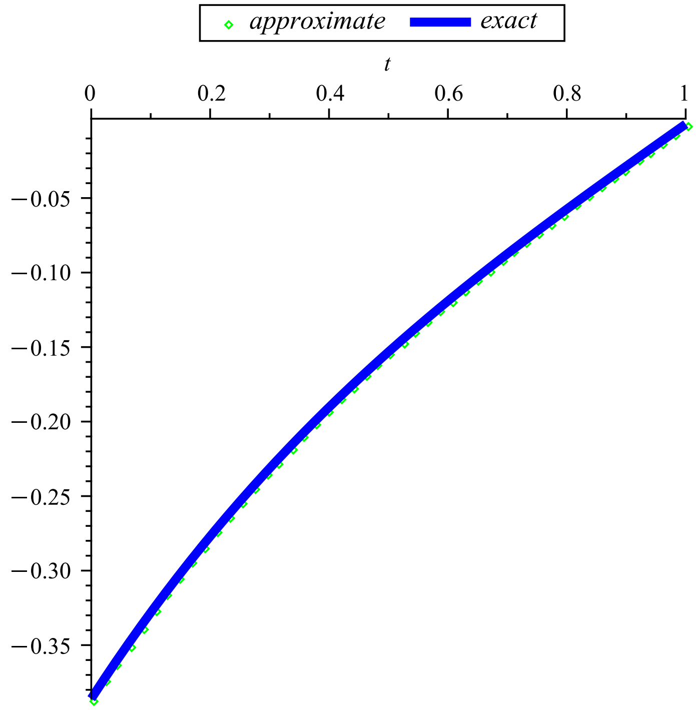

Fig. 4 The graphs of approximated and exact solution $u(t)$ for Example 3.

Table 1 Exact and estimated values of $x(t)$ for Example 3.

t	Analytic $x(t)$	Stated method for $\alpha = 0.9$	The absolute error for $\alpha = 1$	Absolute error in [21]
0.0	1.0000000000	1.0000000000	0.0	0.0000899
0.2	0.7593708976912	0.7593708978173	1.261×10^{-10}	0.0000325
0.4	0.5798975465472	0.5798980476764	5.011292×10^{-7}	0.0000213
0.6	0.4471261037416	0.4471256204474	4.832942×10^{-7}	0.000103
0.8	0.350363853313	0.3503638530515	2.61×10^{-10}	0.0000914
1.0	0.281818070877	0.2818180709000	2.3×10^{-11}	–

to use and straightforward. These are the main advantages of the present method results).

Example 3. Consider the following time invariant problem (see [21])

$$J = \frac{1}{2}\int_0^1 x^2(t) + u^2(t)\, dt,$$
$$s.t. \;\; D_t^\alpha x(t) = -x(t) + u(t),$$
$$x(0) = 1.$$

Our aim is to find $u(t)$ which minimizes the performance index J. For this problem we have the exact solution in the case of $\alpha = 1$ as follows

$$x(t) = \cosh(\sqrt{2}t) + \beta \sinh(\sqrt{2}t),$$
$$u(t) = (1 + \sqrt{2}\beta)\cosh(\sqrt{2}t) + (\sqrt{2} + \beta)\sinh(\sqrt{2}t),$$

where

$$\beta = -\frac{\cosh(\sqrt{2} + \sqrt{2}\sinh 2}{\sqrt{2}\cosh(\sqrt{2}) + \sinh(\sqrt{2})} = -0.98.$$

According to (11) we should have

$${}_0^C D_t^\alpha x(t) = -x(t) - \lambda(t),$$
$${}_t D_1^\alpha \lambda(t) = x(t) - \lambda(t),$$
$$x(0) = 1, \quad \lambda(1) = 0,$$

Also, the following optimal control law may be computed by using $\frac{\partial H}{\partial u} = 0$

$$u^*(t) = -\lambda(t).$$

By choosing $n = 6$ and $\alpha = 0.8$, the following approximated solution can be found

$$x(t) = 1. + 5.020205491t^4 - 4.04713986t^5 + 1.485620981t^2$$
$$-3.010996506t^3 - 1.385929291t + 1.220057256t^6,$$
$$u(t) = -0.385929291 - -0.062492458t^4 + 0.0173924958t^5$$
$$-0.385880666t^2 + 0.204185442t^3$$
$$+0.614070709t - 0.00173880946t^6.$$

The graphs of approximated and exact solution for $x(t)$ and $u(t)$ are plotted respectively in Figs. 3 and 4. In Table 1, exact, estimated value of $x(t)$, the absolute error of stated method for $\alpha = 1$ and the absolute error in [21] are shown.

5. Conclusions

In the present work, we developed an efficient and accurate method for solving a class of fractional optimal control problems, fractional Riccati differential, and Riccati type differential-difference equation by Bezier curves method. The method is computationally attractive, and also reduces the CPU time and the computer memory at the same time while keeping the accuracy of the solution.

Acknowledgments

The authors would like to thank the anonymous reviewers of this paper for their careful reading, constructive comments and nice suggestions which have improved the paper very much.

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